Scribe for Generating functions

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1 Recap:

Generating functions are a way to capture the entire discrete probability distribution of any specific property of the network for each value of the property as a function. The generating function is represented as a polynomial as:

$$G_0(x) = p_0 + p_1 x + p_2 x^2 + \; + p_k x^k$$

2 Powers of Generating functions:

Let $G_0(x)$ denote the degree distribution of the nodes of the network. Then consider the second power of $G_0(x)$.

$$[G_0(x)]^2 = p_0 p_0 x^0 + (p_0 p_1 + p_1 p_0) x^1 + (p_0 p_2 + p_1 p_1 + p_2 p_0) x^2 + \ldots + (\sum p_i p_j) x^k + \ldots$$

where $i + j = k$. Note that the coefficient of every term x^k is the probability that the sum of degrees of two nodes of the network is k. Thus, we get the generating function for the probability distribution of the sum of degrees of two nodes.

In general the k^{th} power of $G_0(x)$ will give the probability distribution of the sum of degrees of k nodes. Further, in the most general sense:

If the distribution of a property x of the object is generated by the generating function $G_0(x)$, then the distribution of the total of x summed over k independent realizations of the object is generated by $[G_0(x)]^k$.

3 Distribution of degree of neighbours:

Choose an edge at random. The probability that such an edge is incident on a vertex of degree k is proportional to kp_k. So the probability distribution of a randomly chosen edge being incident on a vertex of degree k can be given as:

$$\frac{\sum kp_k x^k}{\sum kp_k} = x\frac{\dot{G_0}(x)}{G_0(1)}$$

Thus if we start with at a randomly chosen vertex and follow each of the edges at that vertex to reach its neighbours, then the above function generates the distribution of outgoing edges of the vertices arrived. To generate the distribution of only the remaining outgoing edges, reduce the degree of x by 1. Let this generating function be denoted as $G_1(x)$. Then

$$G_1(x) = \frac{G_0(x)}{G_0(1)} = \frac{1}{z}\dot{G}_0(x)$$

where z is the average degree of the nodes in the network.

4 Distribution of number of second neighbours:

The generating function $G_1(x)$ gave the distribution of degree of neighbours. The k^{th} power of this function will give distribution of the sum of degrees of k nodes. So the following generating function will give the distribution of the number of second neighbours of a node.

$$\sum p_k[G_1(x)]^k = G_0[G_1(x)].$$

The average number of second neighbours will be given by the derivative of the above generating function at $x = 1$.

$$z_2 = \left[\frac{d}{dx}G_0(G_1(x))\right]_{x=1} = G_0(1)G_1(1) = z\frac{\dot{1}}{z}\dot{G}_0(1).$$

5 Application of generating functions to special graphs:

5.1 Graphs following Poisson Degree Distribution

The Poisson degree distribution is a result of binomial distribution as $N \to \infty$. As N becomes large, p = z/N. So we can derive the generating function for Poisson graphs as follows:

$$\mathrm{G}_0(x) = \textstyle\sum_{k=0}^{N} {}^N C_k p^k (1-p)^{N-k} x^k$$
$$\mathrm{G}_0(x) = (1p + px)^N$$
$$\mathrm{G}_0(x) = e^{z(x-1)}$$

The average degree of nodes is:

$$G_0(1) = z\dot{e}^{z(1-1)} = z$$

the probability of a node having degree k is:

$$p_k = \frac{1}{k!}\frac{\dot{d}^k}{dx}G(x)\mid_{x=0}$$
$$p_k = \frac{1}{k!}\frac{\dot{d}^k}{dx}e^{z(x-1)}\mid_{x=0}$$
$$p_k = \frac{z^k}{k!}e^{-z}$$

which is nothing but the probability density function for Poisson distribution.

$$G_1(x) = \frac{G_0(x)}{G_0(1)}$$
$$G_1(x) = \frac{1}{z}\dot{G}_0(x)$$
$$G_1(x) = e^{z(x-1)}$$
$$G_1(x) = G_0(x)$$

So, degree distribution of vertices reached through a randomly chosen edge is the same as the degree distribution of randomly chosen vertices.
The average number of second neighbours is:

$$z_2 = G_0(1)G_1(1) = z^2$$

5.2 Graphs following Exponential degree distribution:

The Exponential probability distribution function is defined as:

$$p_k = (1 - e^{-1/T})e^{-k/T}$$

So, its generating function will be:

$$G_0(x) = (1 - e^{-1/T}) \sum_{k=0}^{\infty} e^{-k/T} x^k$$
$$G_0(x) = (1 - e^{-1/T}) \sum_{k=0}^{\infty} (e^{-1/T} \dot{x})^k$$
$$G_0(x) = \frac{1 - e^{-1/T}}{1 - e^{-1/T} x}$$
$$G_0(x) \simeq (1 - e^{-1/T})(1 + e^{-1/T} x)$$

The average degree of the node will be:

$$G_0(x) = (1 - e^{-1/T})\dot{e}^{-1/T}$$

5.3 Graphs following arbitrarily specified degree distribution

In some cases, we can measure the degree distribution of the real networks and hence can specify it. In that case the generating function will take the form:

$$G_0(x) = \frac{\sigma_k n_k x^k}{\sigma_k n_k}$$

Consider the example of a network of 1000 people each of whom knows between 0 to 5 people. The degree distribution of this acquaintance network is:86,150,363,238,109,54. So,

$$G_0(x) = \frac{86 + 150x + 363x^2 + 238x^3 + 109x^4 + 54x^5}{1000}$$

Then the average degree of the network will be:

$$\mathrm{G}_0(x)|_{x=1} = \tfrac{1}{1000}(150 + 726x + 714x^2 + 436x^3 + 270x^4)|_{x=1}$$
$$\mathrm{G}_0'(x)|_{x=1} = 2.296$$

6 Giant Component:

The probability of an edge coming out of a vertex of degree k is p_k/k. Then after normalization, the generating function for probability distribution of an edge coming out of vertex with degree k is:

$$I_0(x) = \frac{1}{z}\sum \frac{p_k x^k}{k}$$
$$I_0(x) = \frac{1}{z}\sum \frac{p_k}{x^{k-1}}$$
$$I_0(1) = \frac{1}{z}$$

So, arriving at that edge is independent of the probability distribution of the node that is traversed.
Hence arriving at an edge from a random node is equivalent to the probability of arriving at a random node and traversing the edges.

References

[1] M.E.J. Newman, S.H. Strogatz, D.J. Watts *Random graphs with arbitrary degree distributions and their Applications*