Poisson Random Graph

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31-03-2006

1 Introduction

Poisson random graph [1] is a spacial type of random graph model. Erdos and Rényi have discovered this random graph model. As a result this model is also called *E-R graph*.

2 E-R graph

Consider some n number of vertices and connect each pair of vertices with an edge with some *probability* p. In this way the *E-R graph* is modeled as $G_{n,p}$. Actually $G_{n,p}$ is the *ensemble* of all possible graphs with m edges appear with probability $p^m(1-p)^{M-m}$ where $M = n(n-1)/2$. In this model the probability of a vertex having degree k is
$p_k =^n C_k p^k (1-p)^{n-k} \simeq \frac{z^k e^{-z}}{k!}$ where $z = np$
Actually, for large n *binomial distibution* follows *poisson distribution*. This is the reason why this model of graphs is called *poisson random graph*. This type of graphs shows a very important property which is called *phase transition*. In *phase transition* the graph transforms from a low density state, where there are few edges and all components are small to a high-density state where all vertices are joined together in a single *giant component*.

3 Expected size of the giant component

From this model the expected size of the *emerging giant component* can be calculated as follows.

Let u be the fraction of vertices on the graph that does not belong to the *giant component*, which is also the probability that a vertex chosen uniformly at random from the graph is not in the *giant component.* The probability of a vertex not belonging to the giant component is also equal to the probability that none of the vertexs network neighbors belong to the giant component, which is just u^k if the vertex has degree k. So we can write,

$u = \sum_{k=0}^{\infty} p_k u^k$
$= \sum_{k=0}^{\infty} z^k e^{-z} u^k / k!$
$= e^{-z} \sum_{k=0}^{\infty} (zu)^k / k!$
$= e^{-z} e^{zu}$
$= e^{-z(1-u)}$

Let $S = 1 - u$, be the probability of any vertex in the *giant component.* So,
$S = 1 - e^{-zS}$
This is an *open equation* with respect to S. It can be empirically shown that for $z < 1$ its only nonnegative solution is $S = 0$, while for $z > 1$ there is a *non-zero solution*, which is the *size of the giant component.* The *phase transition* occurs at $z = 1$.
We know that
$pn = z$
i.e. $p \propto n^{-1}$ if z is constant and finite.
For large graph as $n \to \infty \quad p \to 0$.
Now the probability of having loop of size $2, 3, \ldots$ is $p^2, p^3, \ldots$. So, the probability of having loops in a large graph following this model tends to 0. So for large graph under this model we can consider it to be a tree. The *random graph* of this model shows one of the principal features of real-world networks i.e. *small world* effect. The mean number of neighbors a distance l away from a vertex in the random graph is z^l. So,
$z^l = n$
$\Rightarrow l = \frac{logn}{logz}$
For real world $n \simeq 6 \times 10^9$. Now according to Milgram's experiment $l \leq 6$.
From this data we get $z \simeq 42.6$, which does not satisfy the real world statistics, as each people in the real world know much much greater than 42 people. Even for smaller values of l we do not get realistic vaule of z. so, we can say that our real life can not be modeled as a *E-R graph* i.e. a *poisson random graph.*

References

[1] "The structure and function of complex networks", M. E. J. Newman, SIAM Review 45, 167-256 (2003).