Epidemics - SIS Model

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I. Herd Immunity

In this section we will see another potentially surprising fact: You do not need to immunize everyone in the population in order to prevent an epidemic.

To see this first recall that it is decreasing in the amount of susceptible individuals in the population. More exactly we saw above that if 1 + βs_t - κ is less than 1 the number of infected individuals will decrease. The social contact structure and the rate of removal, β and κ, are parameters of the social structure of the population and the disease of interest. There is nothing that we can do about κ since it is a function of biology and physiology. (Well, doctors and medical researchers can work on this problem but for the most part we cannot.) We do have some control over the β since it is a function of the population. We can make sick children and workers stay home when they are ill so that they do not infect others in the population. Recall the days of school closings if too many students are sick. (You probably recall those days fondly if you weren't one of those infected.) We also can quarantine people for certain diseases. This would probably happen if a small pox outbreak occurred. In the next section of the course we will spend a great deal of time discussing specifically the contact structure of social populations and the potential effects of public policies designed to deal with epidemics.

But for now we will do something more simple. Let us assume that the good doctors and medical researchers of the world have done their job and found a vaccine for our disease of interest. And that if we immunize a person they are forever immune from catching the disease and thus from passing it on to someone else. Now, from a modelling perspective this essentially means that when a person is immunized they shift directly from the susceptible pool to the recovered pool without going through the infected phase. Instead of the following picture: Susceptible to Infected to Recovered we have: Susceptible to Recovered for those that are immunized. Immunizing people from a disease is costly. There are the direct costs of producing and administering the dosage as well as indirect costs of providing information to the public. Thus we would like to be able to provide safety from disease at the lowest possible cost. What we will see in a moment is that we only have to immunize a fraction (many times a small fraction) of the population in order to succeed in our task. Specifically we need to move enough people from the susceptible pool to the recovered (or in this case immune) pool such that the disease disappears of its own accord. In terms of our model we need to move enough people such that: 1 + βs_0 - κ ¡ 1. In other words we need to lower the epidemic threshold below one. Or to write it more directly the fraction of people that needs to be immunized is such that s_0 ¡ κ/β . Thus you need to immunize 1 - κ/β percent of the susceptible pool. We should see that this makes intuitive sense. If κ is small that means that it takes longer to recover from infection and an infective person has more time to infect people. Thus as κ decreases κ/β increases; you need to inoculate a larger fraction of the population. And as β increases, each infected person contacts more people in a given period, 1 - κ/β increases. Thus again you need to inoculate a larger fraction of the population.

II. SIS Model

Let $i_t = I_t \,/N_t$ be the percent of the population that is currently infected. Define $s_t = S_t \,/N_t$ as the percent of the population that is currently susceptible. We also will define a parameter κ that measures the percent of the population that recovers from a disease each period. Thus if the time to recover is 3 time periods then = 1/3. This means that onethird of the population should recover each period on average. (It may seem that this is a weird assumption since it may be that there are different numbers of people infected in each period. Thus different numbers of people should recover each period. We will show in a moment that if we are in steady state this assumption will hold. If you still have a question on this ask after we have finished this lecture.) We are now ready to write a system of equations that will describe our full SIS model of epidemics:

$I_{t+1} = I_t$ - κI_t + $\alpha\gamma s_t I_t$

$S_{t+1} = S_t$ + κI_t - $\alpha\gamma s_t I_t$

We have κI_t individuals who recover each period and thus leave the infected group and re-enter the susceptible group. And we have $\alpha\gamma s_t I_t$ individuals who enter the infected group and leave the susceptible group. Notice that the change in the infected group always equals the change in the susceptible group when we have a constant population. Thus

$S_t + I_t = S_{T+1} + I_{t+1} = N$ if the population is constant. Equivalently we can write these equations using our proportion state variables:

$i_{t+1} = i_t$ - κi_t + $\alpha\gamma s_t i_t$

$s_{t+1} = s_t$ + κi_t - $\alpha\gamma s_t i_t$

Now, with these equations written we can use them to understand the epidemic threshold in the SIS model. First let us ask the question: when will the number of infected individuals be increasing? Intuitively we can reason through this process by just looking at the equations and our box diagram. If more people flow into the infective state than flow out this means that the level of infectives is increasing. If the opposite is true (more flow out than in) the level of infectives is decreasing. Thus our epidemic threshold is determined by whether κi_t ¿ $\alpha\gamma s_t i_t$ or κi_t ¡ $\alpha\gamma s_t i_t$.

If κ ¿ $\alpha\gamma s_t$ this means that more people leave the infective state than enter it. Thus the level of the disease is decreasing. If κ ¡ $\alpha\gamma s_t$ this means that more people enter the infective state than leave it. Thus the level of the disease is increasing. Now, another way to write this inequality is: $\alpha\gamma s_t/\kappa$. This is the epidemic threshold for the SIS model. If the fraction is greater than one the level of infectives increases (more than one person is infected by each infective) And if it is less than one the level of infectives decreases (fewer than one person is infected by each infective.)

Now one thing different about this model compared to our previous one is that one of our state variables enters the equation for the epidemic threshold, s_t . If $\alpha\gamma s_t/\kappa$ ¿ 1 the disease spreads and the number of susceptible individuals decreases. Thus the fraction gets smaller in the next period. If $\alpha\gamma s_t/\kappa$ ¡ 1 the disease begins to die out and the number of susceptibles increases. Thus the fraction gets bigger in the next period. For those of you who have taken economics courses this reasoning may sound familiar. It may sound like a process that is working its way to equilibrium. And that is exactly what we will see in many cases here. The lack of susceptible individuals slows the epidemic when there are many infective individuals; and many susceptible individuals increases the spread of the epidemic when there are few infectives. The number of available hosts (or susceptible individuals) introduces negative feedback into the SIS model.

What would be the case where the disease reaches equilibrium? By this I mean the number of susceptibles and infectives is in steady state; both are constant proportions of the population. This would be the case if $\alpha\gamma s_t/\kappa = 1$ or $s_t = \kappa/\alpha\gamma$. This would define a steady state of the system where st and it are constant in all periods.

A. Steady State

We are now ready to analyze our first full model of epidemics a little more formally by looking at the steady state of our system of equations. Again, for those of you who have taken previous economics courses you may think of a steady state as an equilibrium: a situation where a system is not changing. In this model we will think of a steady state as a state where our state variables, I_t and S_t , do not change from period to period; they are constant. To find the steady state we need to find a solution to our system of equation above where $S_t = S_{t+1}$ and $I_t = I_{t+1}$. One way to do this is to drop the time subscript on our equations above and solve for s and i. Thus we are looking for a solution to the following set of equations:

$i = i - \kappa i + \alpha\gamma si$

$s = s + \kappa i - \alpha\gamma si$

to solve these equations first rewrite $i = i - \kappa i + \alpha\gamma si$ as: $\kappa i = \alpha\gamma si$, Which can then be written as: $s_t = \kappa/\alpha\gamma$

This is the steady state value of the proportion of the population that is susceptible. It is the same equation as we found above from our intuitive understanding of the epidemic. And since we know that $s_t + i_t = 1$, we know that the steady state value of the infected proportion of the population is $i_t = 1 - s_t$ or : i=1- $\kappa/\alpha\gamma$.

Note the intuitive properties of these equations. As the number of contacts of an infected person or the transmission probability increases the number of susceptible individuals decreases (and the number of infected individuals increases). And as the time to recover increases (meaning κ decreases) the number of susceptible individuals decreases (the number of infected people increases.)